

Dual wavefunction of the symplectic ice

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Abstract

The wavefunction of the free-fermion six-vertex model was found to give a natural realization of the Tokuyama combinatorial formula for the Schur polynomials by Bump-Brubaker-Friedberg. Recently, we studied the correspondence between the dual version of the wavefunction and the Schur polynomials, which gave rise to another combinatorial formula. In this paper, we extend the analysis to the reflecting boundary condition, and show the exact correspondence between the dual wavefunction and the symplectic Schur functions. This gives a dual version of the integrable model realization of the symplectic Schur functions by Ivanov. We also generalize to the correspondence between the wavefunction, the dual wavefunction of the six-vertex model and the factorial symplectic Schur functions by the inhomogeneous generalization of the model.

Mathematics Subject Classification. 05E05, 05E10, 16T25, 16T30, 17B37.

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1 Introduction

The study of the connections between integrable lattice models [1, 2, 3, 4] and combinatorial representation theory of symmetric polynomials is an active area of research in these years.

The main actor is the wavefunction in this research, which is constructed from the R -matrices satisfying the Yang-Baxter relation. The wavefunction of the most famous six-vertex model [5, 6] whose R -matrix comes from the Drinfeld-Jimbo quantum group [7, 8] $U_q(sl_2)$ representation, and its $q = 0$ five-vertex model degeneration, has turned out to give a representation of the Grothendieck polynomials and its quantum group deformation (see [9, 10, 11, 12, 13, 14, 15, 16, 17] for example). Based on this correspondence, various algebraic combinatorial identities have been discovered and proved. It seems that many of the

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identities themselves are hard to be discovered without the power of quantum integrability. These developments may shed new light in the world of Schubert calculus [18, 19, 20, 21, 22, 23, 24, 25]. For example, the notion of excited Young diagram [21] in the field of Schubert calculus is essentially equivalent to the wavefunction of certain integrable five-vertex models. Translating into the language of quantum integrable models can give new insights. For example, there is a recent development on the investigation of the Littlewood-Richardson coefficients from the point of view of quantum integrability [13].

The object treated in this paper is based on another way of direction of the developments uncovered by number theorists. Bump, Brubaker and Friedberg found [26] that a certain kind of free-fermion six-vertex model gives a natural realization of the Tokuyama formula [27], which gives a deformation of the Weyl character formula. The free-fermion six-vertex model can be regarded as a gauge-deformed version of the Felderhof free-fermion model [28], whose underlying quantum group symmetry can be explained either as an exotic roots of unity finite-dimensional highest weight representation [29, 30]. Another explanation is that the free-fermion six-vertex model corresponds to the simplest case of the Perk-Schultz model [31] which is a representation of quantum supergroup. The latter formulation recently gave rise to the Yang-Baxter equation [32] for the metaplectic ice [33]. As for the domain wall boundary partition function which is a special class of partition functions, it was evaluated in [34, 35] by using the Izergin-Korepin technique in the past and factorization phenomena was observed.

However, it was only found in recent years that the free-fermion model has rich mathematical structures related with the combinatorial representation theory of Schur polynomials. One of the striking facts found [26] was that the Tokuyama formula [27], which is a one-parameter deformation of the Weyl character formula, is naturally realized as the wavefunction of the free-fermion model. The wavefunction is the most fundamental object in the statistical physical aspects of quantum integrable models, since it becomes the Bethe eigenvectors of the corresponding one-dimensional spin chain when the Bethe ansatz equation is imposed on the spectral parameters.

The Tokuyama formula for the Schur polynomials can be understood as a consequence of the evaluation of the wavefunction in two ways. One by expressing it as a product of a one-parameter deformation of the Vandermonde determinant and the Schur polynomials, and another one by making a microscopic analysis and derive an expression using the strict Gelfand-Tsetlin pattern. The Tokuyama formula is a consequence of the two evaluations for the same object. This understanding [26] opened a new doorway to the combinatorial representation theory of symmetric polynomials via the free-fermion model. One of the progresses after this work is the construction of variations of the Tokuyama-type formulas by changing boundary conditions. Several Tokuyama-type formulas for types other than type A was obtained by Okada [36] and Hamel-King [37, 38] earlier, and recently by using methods of analytic number theory [39] and non-intersecting lattice paths [40, 41, 42].

They were investigated from the viewpoint of quantum integrability in [43, 44, 45, 46] where local objects and relations such as the L -operators, K -matrix and the Yang-Baxter relation are extensively used. For example, the correspondence between the wavefunction under the half-turn boundary condition and the symplectic Schur functions was obtained by Ivanov in [43].

On the other hand, we studied the dual wavefunction of the free-fermion model in a recent paper [47]. We gave the exact correspondence between the dual wavefunction and the Schur

polynomials. which includes the special case $t = 1$ of the deformation parameter [26, 48], which was obtained by transforming the original wavefunction to the dual wavefunction by symmetry arguments. We gave two proofs for the correspondence. One proof used transformation of the statement of the theorem to an equivalent statement so that one can use the arguments given in [26]. Another proof is a modern statistical mechanical approach, which combines the matrix product method [49, 50] and the Izergin-Korepin method of analysis on the domain wall boundary partition function [51, 52]. The exact correspondence with the Schur polynomials, together with a microscopic analysis of the dual wavefunction gave rise to a dual version of the Tokuyama-type formula for the Schur polynomials.

In this paper, we combine these two directions of progresses, and extend the study of the dual wavefunction to the free-fermion model under the reflecting boundary condition. We give the exact correspondence between the dual wavefunction and the symplectic Schur functions. We prove the correspondence by extending the argument used in the first proof in [47] to the reflecting boundary condition, so that one can use the arguments by Brubaker-Bump-Friedberg [26] and Ivanov [43]. This gives a dual version of type C Tokuyama formula. We also generalize the Theorems by Ivanov and the main result in this paper to give the exact correspondence between the wavefunction, dual wavefunction and the factorial symplectic Schur functions by using two types of L -operators and the K -matrix of the inhomogeneous six-vertex model. This is a symplectic analogue of the work of Bump-McNamara-Nakasuji [48] which they established the Tokuyama-type formulas for factorial Schur functions. Recently, the Tokuyama-type formulas for factorial characters were obtained by using non-intersecting lattice paths by Hamel-King [41, 42] which is worthwhile to investigate the relation and the results in this paper in the future.

This paper is organized as follows. We introduce the free-fermion model in section 2 and review the relation between the wavefunction under the reflecting boundary condition and the symplectic Schur functions in section 3. In section 4, we introduce the dual wavefunction, and prove the relation with the symplectic Schur functions, which can be regarded as a dual version of type C Tokuyama formula. In section 5, we generalize the correspondence and give the exact relation between the wavefunction, dual wavefunction of the inhomogeneous free-fermion model and the factorial symplectic Schur functions. Section 6 is devoted to the conclusion.

2 Free-fermion model

We introduce the free-fermion model in this section, and review the results on the relation between the wavefunction and the symplectic Schur functions in the next section. We use the L -operator in [26] which is best suited for the study of the combinatorics of the Schur polynomials, since the Tokuyama formula is exactly realized as the wavefunction constructed from this L -operator. More generic or gauge-transformed ones can be found in [29, 30, 35] for example. We also use the terminology of the quantum inverse scattering method or the algebraic Bethe ansatz, which is one of the most fundamental methods for the analysis of quantum integrable models.

The most fundamental objects in integrable lattice models are the R -matrix and the

L -operator. For the case of the free-fermion model we consider, the R -matrix is given by

$$R_{ab}(z, t) = \begin{pmatrix} 1 + tz^{-1} & 0 & 0 & 0 \\ 0 & t(1 - z^{-1}) & t + 1 & 0 \\ 0 & (t + 1)z^{-1} & z^{-1} - 1 & 0 \\ 0 & 0 & 0 & z^{-1} + t \end{pmatrix}, \quad (2.1)$$

acting on the tensor product $W_a \otimes W_b$ of the complex two-dimensional space W_a . Let us denote the orthonormal basis of W_a and its dual as $\{|0\rangle_a, |1\rangle_a\}$ and $\{{}_a\langle 0|, {}_a\langle 1|\}$, and the matrix elements of the R -matrix as ${}_a\langle \gamma|{}_b\langle \delta|R_{ab}(z, t)|\alpha\rangle_a|\beta\rangle_b = [R_{ab}(z, t)]_{\alpha\beta}^{\gamma\delta}$. The matrix elements of the R -matrix are explicitly given as

$${}_a\langle 0|{}_b\langle 0|R_{ab}(z, t)|0\rangle_a|0\rangle_b = 1 + tz^{-1}, \quad (2.2)$$

$${}_a\langle 0|{}_b\langle 1|R_{ab}(z, t)|0\rangle_a|1\rangle_b = t(1 - z^{-1}), \quad (2.3)$$

$${}_a\langle 0|{}_b\langle 1|R_{ab}(z, t)|1\rangle_a|0\rangle_b = t + 1, \quad (2.4)$$

$${}_a\langle 1|{}_b\langle 0|R_{ab}(z, t)|0\rangle_a|1\rangle_b = (t + 1)z^{-1}, \quad (2.5)$$

$${}_a\langle 1|{}_b\langle 0|R_{ab}(z, t)|1\rangle_a|0\rangle_b = z^{-1} - 1, \quad (2.6)$$

$${}_a\langle 1|{}_b\langle 1|R_{ab}(z, t)|1\rangle_a|1\rangle_b = z^{-1} + t. \quad (2.7)$$

The L -operator of the free-fermion model is given by

$$L_{aj}(z, t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & (t + 1)z^{-1} & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix}, \quad (2.8)$$

acting on the tensor product $W_a \otimes \mathcal{F}_j$ of the space W_a and the two-dimensional Fock space at the j th site \mathcal{F}_j . We also denote the orthonormal basis of \mathcal{F}_j and its dual as $\{|0\rangle_j, |1\rangle_j\}$ and $\{{}_j\langle 0|, {}_j\langle 1|\}$, and the matrix elements of the L -operator as ${}_a\langle \gamma|{}_j\langle \delta|L_{aj}(z, t)|\alpha\rangle_a|\beta\rangle_j = [L_{aj}(z, t)]_{\alpha\beta}^{\gamma\delta}$. The matrix elements of the L -operator are explicitly written as (see Figure 1 for a pictorial description)

$${}_a\langle 0|{}_j\langle 0|L_{aj}(z, t)|0\rangle_a|0\rangle_j = 1, \quad (2.9)$$

$${}_a\langle 0|{}_j\langle 1|L_{aj}(z, t)|0\rangle_a|1\rangle_j = t, \quad (2.10)$$

$${}_a\langle 0|{}_j\langle 1|L_{aj}(z, t)|1\rangle_a|0\rangle_j = 1, \quad (2.11)$$

$${}_a\langle 1|{}_j\langle 0|L_{aj}(z, t)|0\rangle_a|1\rangle_j = (t + 1)z^{-1}, \quad (2.12)$$

$${}_a\langle 1|{}_j\langle 0|L_{aj}(z, t)|1\rangle_a|0\rangle_j = z^{-1}, \quad (2.13)$$

$${}_a\langle 1|{}_j\langle 1|L_{aj}(z, t)|1\rangle_a|1\rangle_j = z^{-1}. \quad (2.14)$$

The R -matrices and the L -operators have origins in statistical physics, and $|0\rangle$ or its dual $\langle 0|$ can be regarded as a hole state, while $|1\rangle$ or its dual $\langle 1|$ can be interpreted as a particle state from the point of view of statistical physics. We use the terms hole states and particle states to describe states constructed from $|0\rangle$, $\langle 0|$, $|1\rangle$ and $\langle 1|$ from now on since they are convenient for the description of the states. We also remark that in the language of the quantum inverse scattering method, the Fock spaces W_a and \mathcal{F}_j are usually called as the auxiliary and quantum spaces, respectively.

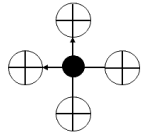
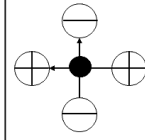
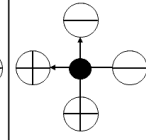
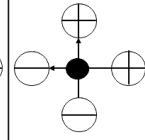
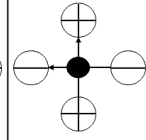
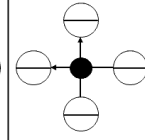
					
1	t	1	$(t+1)z^{-1}$	z^{-1}	z^{-1}

Figure 1: The first L -operator (2.8). The (dual) state $|0\rangle$ ($\langle 0|$) is represented as \oplus , while the (dual) state $|1\rangle$ ($\langle 1|$) is represented as \ominus , following the pictorial description of [26].

The R -matrix (2.1) and L -operator (2.8) satisfy the Yang-Baxter relation

$$R_{ab}(z_1/z_2, t)L_{aj}(z_1, t)L_{bj}(z_2, t) = L_{bj}(z_2, t)L_{aj}(z_1, t)R_{ab}(z_1/z_2, t), \quad (2.15)$$

acting on $W_a \otimes W_b \otimes \mathcal{F}_j$. We remark that this RLL relation (2.15) can be regarded as a special case of the generalized Yang-Baxter relation for a more general R -matrix [29, 30, 35]. The R -matrix (2.1) and the L -operator (2.8) in this section can be regarded as different specializations of the general R -matrix from this viewpoint. One of the advantages of the point of view from the quantum group is that one can systematically generalize the Felderhof model to higher-dimensional representations [30].

To realize the Tokuyama formula for Schur polynomials, it was enough to use the L -operator (2.8). To deal with symplectic Schur functions, one needs more objects. We introduce the second L -operator (see Figure 2)

$$\tilde{L}_{aj}(z, t) = \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & tz & 1 & 0 \\ 0 & (t+1)z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.16)$$

whose matrix elements are explicitly given by

$${}_a\langle 0|_j \langle 0| \tilde{L}_{aj}(z, t) |0\rangle_a |0\rangle_j = z, \quad (2.17)$$

$${}_a\langle 0|_j \langle 1| \tilde{L}_{aj}(z, t) |0\rangle_a |1\rangle_j = tz, \quad (2.18)$$

$${}_a\langle 0|_j \langle 1| \tilde{L}_{aj}(z, t) |1\rangle_a |0\rangle_j = 1, \quad (2.19)$$

$${}_a\langle 1|_j \langle 0| \tilde{L}_{aj}(z, t) |0\rangle_a |1\rangle_j = (t+1)z, \quad (2.20)$$

$${}_a\langle 1|_j \langle 0| \tilde{L}_{aj}(z, t) |1\rangle_a |0\rangle_j = 1, \quad (2.21)$$

$${}_a\langle 1|_j \langle 1| \tilde{L}_{aj}(z, t) |1\rangle_a |1\rangle_j = 1. \quad (2.22)$$

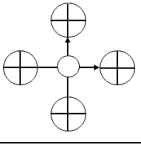
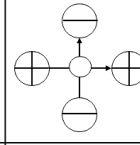
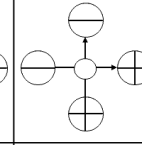
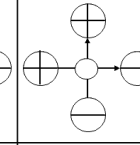
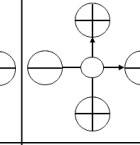
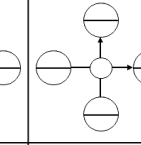
					
z	tz	1	$(t+1)z$	1	1

Figure 2: The second L -operator (2.16). We follow the pictorial description of [43] for this second L -operator.

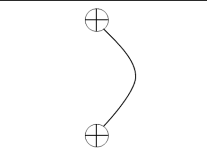
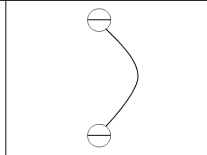
	
tz	z^{-1}

Figure 3: The K -matrix (2.23). We follow the pictorial description of [43] for the K -matrix.

We also introduce the K -matrix acting on the auxiliary space W_a (see Figure 3)

$$K_a(z, t) = \begin{pmatrix} tz & 0 \\ 0 & z^{-1} \end{pmatrix}. \quad (2.23)$$

The K -matrix is used when partition functions of integrable lattice models under reflecting boundary conditions are considered. The matrix elements are explicitly given by

$${}_a\langle 0|K_a(z, t)|0\rangle_a = tz, \quad (2.24)$$

$${}_a\langle 1|K_a(z, t)|1\rangle_a = z^{-1}. \quad (2.25)$$

From the L -operator, we construct two monodromy matrices as products of L -operators

$$T_a(z) = L_{aM}(z, t) \cdots L_{a1}(z, t), \quad (2.26)$$

and

$$\tilde{T}_a(z) = \tilde{L}_{a1}(z, t) \cdots \tilde{L}_{aM}(z, t), \quad (2.27)$$

$$A(z)$$

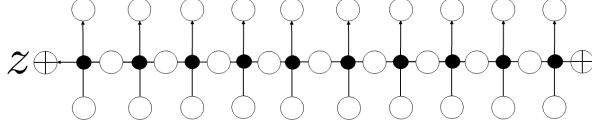


Figure 4: The A -operator $A(z)$ (2.28), which is a matrix element of the monodromy matrix $T_a(z)$. The A -operator is $2^M \times 2^M$ matrix-valued. Both the leftmost and rightmost state on the horizontal line (auxiliary space) are fixed as \oplus .

$$B(z)$$

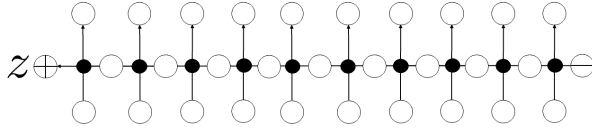


Figure 5: The B -operator $B(z)$ (2.29). The leftmost state on the horizontal line is fixed as \oplus , whereas the rightmost state is fixed as \ominus .

$$\tilde{A}(z)$$

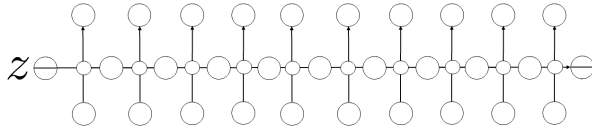


Figure 6: The second A -operator $\tilde{A}(z)$ (2.30). Both the leftmost and rightmost state on the horizontal line are fixed as \ominus .

$\tilde{B}(z)$

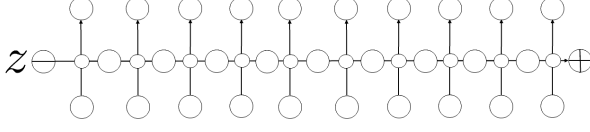


Figure 7: The second B -operator $\tilde{B}(z)$ (2.31). The leftmost state on the horizontal line is fixed as \ominus , whereas the rightmost state is fixed as \oplus .

which act on $W_a \otimes (\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M)$.

In the language of quantum inverse scattering method, the matrix elements of the monodromy matrices with respect to the auxiliary space are called as $ABCD$ operators. In this paper, we consider two types of A - and B -operators

$$A(z) = {}_a\langle 0|T_a(z)|0\rangle_a, \quad (2.28)$$

$$B(z) = {}_a\langle 0|T_a(z)|1\rangle_a, \quad (2.29)$$

and

$$\tilde{A}(z) = {}_a\langle 1|\tilde{T}_a(z)|1\rangle_a, \quad (2.30)$$

$$\tilde{B}(z) = {}_a\langle 0|\tilde{T}_a(z)|1\rangle_a. \quad (2.31)$$

See Figures 4, 5, 6 and 7 for a graphical description of these two types of A - and B -operators.

$\mathcal{B}(z)$

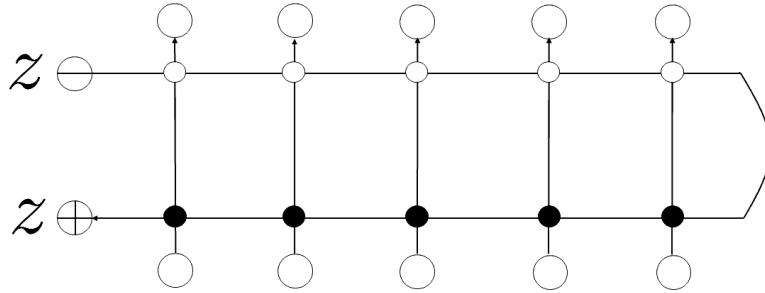


Figure 8: The double row B -operator $\mathcal{B}(z)$ (the left hand side of (2.33)).

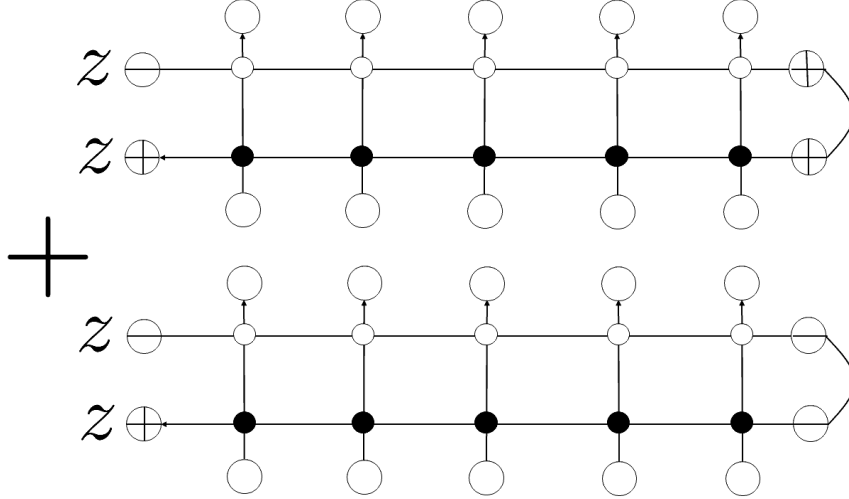


Figure 9: The decomposition of the double row B -operator $\mathcal{B}(z)$ (the right hand side of (2.33)). The top figure where the rightmost states are fixed as \oplus represents the term $\tilde{B}(z)_a \langle 0|K(z,t)|0\rangle_a A(z) = tz\tilde{B}(z)A(z)$. The bottom figure where the rightmost states are fixed as \oplus represents the term $\tilde{A}(z)_a \langle 1|K(z,t)|1\rangle_a B(z) = z^{-1}\tilde{A}(z)B(z)$.

By using these four monodromy operators and K -matrix, we introduce the following double row B -operator [53] which we use to construct the wavefunctions under reflecting boundary.

$$\mathcal{B}(z) = \tilde{B}(z)_a \langle 0|K(z,t)|0\rangle_a A(z) + \tilde{A}(z)_a \langle 1|K(z,t)|1\rangle_a B(z) \quad (2.32)$$

$$= tz\tilde{B}(z)A(z) + z^{-1}\tilde{A}(z)B(z). \quad (2.33)$$

See Figures 8 and 9 for pictorial descriptions of (2.33).

3 Wavefunction and symplectic Schur functions

We introduce the wavefunction which is a special class of partition functions, and review the relation with the symplectic Schur functions defined below.

Definition 3.1. The symplectic Schur functions is defined to be the following determinant:

$$sp_{\lambda}(\{z\}_N) = \frac{\det_N(z_j^{\lambda_k + N - k + 1} - z_j^{-\lambda_k - N + k - 1})}{\det_N(z_j^{N - k + 1} - z_j^{-N + k - 1})}, \quad (3.1)$$

where $\{z\}_N = \{z_1, \dots, z_N\}$ is a set of variables and λ denotes a Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ with weakly decreasing non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$.

Before introducing the wavefunction, we first define the arbitrary N -particle state $|\Psi(z_1, \dots, z_N)\rangle$ with N spectral parameters $\{z\}_N = \{z_1, \dots, z_N\}$ as a state constructed by

a multiple action of the double row B -operator on the vacuum state $|\Omega\rangle := |0^M\rangle := |0\rangle_1 \otimes \cdots \otimes |0\rangle_M$

$$|\Psi(z_1, \dots, z_N)\rangle = \mathcal{B}(z_1) \cdots \mathcal{B}(z_N)|\Omega\rangle. \quad (3.2)$$

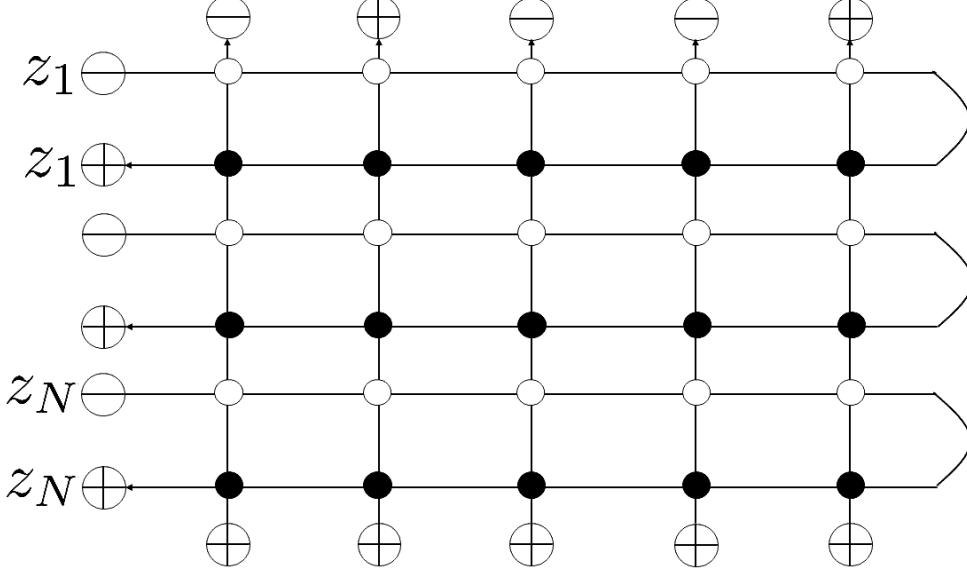


Figure 10: The wavefunction $\langle x_1 \cdots x_N | \Psi(z_1, \dots, z_N) \rangle$ for the case $M = 5$, $N = 3$, $(x_1, x_2, x_3) = (2, 3, 5)$.

Next, we introduce the wavefunction $\langle x_1 \cdots x_N | \Psi(z_1, \dots, z_N) \rangle$ as the overlap between an arbitrary off-shell N -particle state $|\Psi(z_1, \dots, z_N)\rangle$ and the (normalized) state with an arbitrary particle configuration $|x_1 \cdots x_N\rangle$ ($1 \leq x_1 < \cdots < x_N \leq M$), where x_j denotes the positions of the particles. The particle configurations are explicitly defined as

$$\langle x_1 \cdots x_N | = \langle \Omega | \prod_{j=1}^N \sigma_{x_j}^+, \quad (3.3)$$

where $\langle \Omega | := \langle 0^M | := {}_1\langle 0 | \otimes \cdots \otimes {}_M\langle 0 |$. Here, we define σ^+ and σ^- as operators acting on the basis elements as

$$\sigma^+|1\rangle = |0\rangle, \quad \sigma^+|0\rangle = 0, \quad \langle 0|\sigma^+ = \langle 1|, \quad \langle 1|\sigma^+ = 0, \quad (3.4)$$

$$\sigma^-|0\rangle = |1\rangle, \quad \sigma^-|1\rangle = 0, \quad \langle 1|\sigma^- = \langle 0|, \quad \langle 0|\sigma^- = 0. \quad (3.5)$$

The subscript j of σ_j^+ or σ_j^- indicates that the operator acts on the space \mathcal{F}_j as σ^+ or σ^- , and as an identity on the other spaces.

The following correspondence between the wavefunction of the Felderhof model under reflecting boundary and the symplectic Schur functions was proved by Ivanov.

Theorem 3.2. [43] *The wavefunction $\langle x_1 \cdots x_N | \Psi(z_1, \dots, z_N) \rangle$ is expressed by the symplectic Schur functions as*

$$\langle x_1 \cdots x_N | \Psi(z_1, \dots, z_N) \rangle = \prod_{j=1}^N z_j^{j-1-N} (1 + tz_j^2) \prod_{1 \leq j < k \leq N} (1 + tz_j z_k) (1 + tz_j z_k^{-1}) sp_{\lambda}(\{z\}_N). \quad (3.6)$$

Here the Young diagram for the symplectic Schur functions correspond to the particle configuration under the relation $\lambda_j = x_{N-j+1} - N + j - 1$, $j = 1, \dots, N$.

The above theorem means that the product of a deformation of Weyl's denominator and the symplectic Schur functions, which is an irreducible character of the symplectic group $Sp(2n, \mathbb{C})$, can be expressed as a wavefunction of the free-fermion six-vertex model under the reflecting boundary condition. The wavefunction offers an explicit description in terms the Proctor pattern, and hence this result can be regarded as a type C version of the Tokuyama formula.

4 Dual wavefunction

We now introduce the dual wavefunction, and study the exact relation between it and the symplectic Schur functions. The strategy of the proof of the correspondence given here can be regarded as the symplectic version of our proof of the correspondence between the dual wavefunction without reflecting boundary and the Schur polynomials [47]. We transform the statement of the theorem into another equivalent one which enables us to use the arguments of Bump-Brubaker-Friedberg [26] and Ivanov [43].

Before defining the dual wavefunction, we introduce another type of arbitrary dual N -hole state $\langle \Phi(z_1, \dots, z_N) |$ by a multiple action of the double row B -operator on the dual particle occupied state $\langle 1 \cdots M | := \langle 1^M | := {}_1 \langle 1 | \otimes \cdots \otimes_M \langle 1 |$

$$\langle \Phi(z_1, \dots, z_N) | = \langle 1 \cdots M | \mathcal{B}(z_1) \cdots \mathcal{B}(z_N). \quad (4.1)$$

It is convenient to introduce a notation for the state with an arbitrary hole configuration $|\overline{x_1} \cdots \overline{x_N}\rangle$ ($1 \leq \overline{x_1} < \cdots < \overline{x_N} \leq M$), where $\overline{x_j}$ denotes the positions of holes. Explicitly,

$$|\overline{x_1} \cdots \overline{x_N}\rangle = \prod_{j=1}^N \sigma_{x_j}^+ (|1\rangle_1 \otimes \cdots \otimes |1\rangle_M). \quad (4.2)$$

The dual wavefunction $\langle \Phi(z_1, \dots, z_N) | \overline{x_1} \cdots \overline{x_N} \rangle$ is defined as the overlap between the arbitrary dual N -hole state $\langle \Phi(z_1, \dots, z_N) |$ and hole configurations $|\overline{x_1} \cdots \overline{x_N}\rangle$ (see Figure 11 for an example of a graphical description of the dual wavefunction).

We show the following relation between the dual wavefunction and the symplectic Schur functions.

Theorem 4.1. *The dual wavefunction $\langle \Phi(z_1, \dots, z_N) | \overline{x_1} \cdots \overline{x_N} \rangle$ can be expressed by the symplectic Schur functions as*

$$\begin{aligned} & \langle \Phi(z_1, \dots, z_N) | \overline{x_1} \cdots \overline{x_N} \rangle \\ &= t^{N(M-N)} \prod_{j=1}^N z_j^{j-1-N} (1 + tz_j^2) \prod_{1 \leq j < k \leq N} (1 + tz_j z_k) (1 + tz_j z_k^{-1}) sp_{\overline{\lambda}}(\{tz\}_N). \end{aligned} \quad (4.3)$$

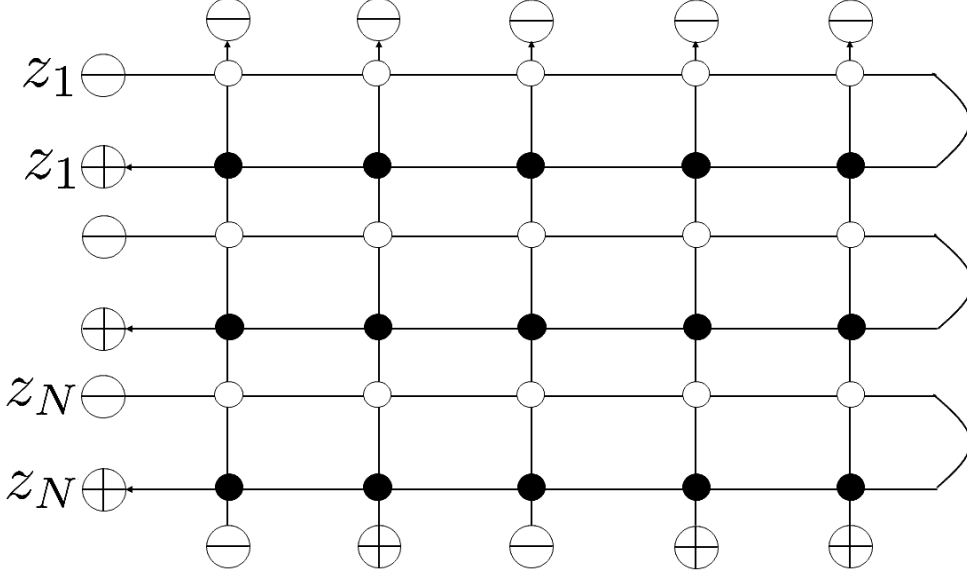


Figure 11: The dual wavefunction $\langle \Phi(z_1, \dots, z_N) | \overline{x_1} \cdots \overline{x_N} \rangle$ for the case $M = 5$, $N = 3$, $(\overline{x_1}, \overline{x_2}, \overline{x_3}) = (1, 2, 4)$.

Here the Young diagram for the symplectic Schur functions correspond to the hole configuration under the relation $\overline{\lambda}_j = \overline{x_{N-j+1}} - N + j - 1$, $j = 1, \dots, N$, and the symmetric variables are $\{tz\}_N = \{tz_1, \dots, tz_N\}$.

The result of Theorem 4.1 resembles the case for the dual wavefunction without reflecting boundary which gives the Schur polynomials. There is again a factor $t^{N(M-N)}$ which depends on the number of sites M and the number of particles N in the right hand side of (4.3). Also, the symmetric variables of the symplectic Schur functions are $\{tz\}_N$, not simply $\{z\}_N$.

It seems difficult to directly prove (4.3) itself by using the argument by Ivanov [43], which extends the one for the case without reflecting boundary by Bump-Brubaker-Friedberg [26] to the symplectic ice. As in the case of the paper which we proved the correspondence between the dual wavefunction without reflecting boundary and the Schur polynomials, the key is to transform the statement (4.3) to another equivalent one, which enables us to give a proof by using the argument by Ivanov.

Proof. We transform (4.3) as follows. First, by rescaling each z_j to $t^{-1}z_j$, we have

$$\begin{aligned} & \langle \Phi(t^{-1}z_1, \dots, t^{-1}z_N) | \overline{x_1} \cdots \overline{x_N} \rangle \\ &= t^{MN} \prod_{j=1}^N z_j^{j-1-N} (1 + t^{-1}z_j^2) \prod_{1 \leq j < k \leq N} (1 + t^{-1}z_j z_k) (t^{-1} + z_j z_k^{-1}) sp_{\overline{\lambda}}(\{z\}_N). \end{aligned} \quad (4.4)$$

We further rewrite (4.4) in the following form.

$$\begin{aligned}
& t^N \langle 1 \cdots M | \left(\frac{\mathcal{B}(t^{-1}z_1)}{t^{M+1}} \right) \cdots \left(\frac{\mathcal{B}(t^{-1}z_N)}{t^{M+1}} \right) | \overline{x_1} \cdots \overline{x_N} \rangle \\
&= \prod_{j=1}^N z_j^{j-1-N} (1 + t^{-1}z_j^2) \prod_{1 \leq j < k \leq N} (1 + t^{-1}z_j z_k) (t^{-1} + z_j z_k^{-1}) sp_{\overline{\lambda}}(\{z\}_N). \tag{4.5}
\end{aligned}$$

For giving a proof, it is convenient to introduce the following rescaled L -operators and K -matrix

$$L'(z, t) = \frac{1}{t} L(t^{-1}z, t) = \begin{pmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & 1 & t^{-1} & 0 \\ 0 & (t+1)z^{-1} & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix}, \tag{4.6}$$

$$\tilde{L}'(z, t) = \tilde{L}(t^{-1}z, t) = \begin{pmatrix} t^{-1}z & 0 & 0 & 0 \\ 0 & z & 1 & 0 \\ 0 & (1+t^{-1})z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.7}$$

$$K'(z, t) = \frac{1}{t} K(t^{-1}z) = \begin{pmatrix} t^{-1}z & 0 \\ 0 & z^{-1} \end{pmatrix}. \tag{4.8}$$

We denote the two types of the A - and B -operators, the double row B -operator constructed from these rescaled L -operators and K -matrix $L'(z, t)$, $\tilde{L}'(z, t)$, $K'(z, t)$ as $A'(z)$, $B'(z)$, $\tilde{A}'(z)$, $\tilde{B}'(z)$ and $\mathcal{B}'(z)$ respectively. The double row B operator is now constructed from the two types of the A - and B -operators as

$$\mathcal{B}'(z) = \tilde{B}'(z)_a \langle 0 | K'(z, t) | 0 \rangle_a A'(z) + \tilde{A}'(z)_a \langle 1 | K'(z, t) | 1 \rangle_a B'(z) \tag{4.9}$$

$$= t^{-1} z \tilde{B}'(z) A'(z) + z^{-1} \tilde{A}'(z) B'(z). \tag{4.10}$$

Using these rescaled objects, (4.5) can be expressed as

$$\begin{aligned}
& t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle \\
&= \prod_{j=1}^N z_j^{j-1-N} (1 + t^{-1}z_j^2) \prod_{1 \leq j < k \leq N} (1 + t^{-1}z_j z_k) (t^{-1} + z_j z_k^{-1}) sp_{\overline{\lambda}}(\{z\}_N). \tag{4.11}
\end{aligned}$$

Instead of proving (4.3), we show (4.11) since this is equivalent to (4.3) and is the expression which one can use the argument given in Ivanov [43].

We first show the following lemma.

Lemma 4.2.

$$\begin{aligned}
& \prod_{j=1}^N z_j^{N+1-j} (1 + t^{-1}z_j^2)^{-1} \prod_{1 \leq j < k \leq N} (1 + t^{-1}z_j z_k)^{-1} (t^{-1} + z_j z_k^{-1})^{-1} \\
& \times t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle \tag{4.12}
\end{aligned}$$

does not depend on t .

Proof. We follow along the lines of the proof by Ivanov [43]. It is enough to prove the following properties for

$$t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle:$$

1. $t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle$ is a polynomial of $t' := t^{-1}$ with its highest degree at most N^2 .
2. $t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle / D'$, where $D' := \prod_{j=1}^N z_j^{j-1-N} (1+t' z_j^2) \prod_{1 \leq j < k \leq N} (1+t' z_j z_k)(t' + z_j z_k^{-1})$, is invariant under any permutation of z_1, \dots, z_N .
3. $t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle / D'$ is invariant under $z_i \longleftrightarrow z_i^{-1}$.

We first show $\deg_{t'}(t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle) \leq N^2$ by induction on N . We use the following properties for the matrix elements of the A - and B -operators, which can easily be seen from the matrix elements of the L -operators.

$$0 \leq \deg_{t'}(t \langle \overline{x_1} \cdots \overline{x_N} | A'(z) | \overline{y_1} \cdots \overline{y_N} \rangle) \leq N-1, \quad N \geq 1, \quad (4.13)$$

$$0 \leq \deg_{t'}(t \langle \overline{x_1} \cdots \overline{x_N} | B'(z) | \overline{y_1} \cdots \overline{y_{N+1}} \rangle) \leq N, \quad N \geq 0, \quad (4.14)$$

$$0 \leq \deg_{t'}(\langle \overline{x_1} \cdots \overline{x_N} | \tilde{A}'(z) | \overline{y_1} \cdots \overline{y_N} \rangle) \leq N, \quad N \geq 0, \quad (4.15)$$

$$0 \leq \deg_{t'}(\langle \overline{x_1} \cdots \overline{x_N} | \tilde{B}'(z) | \overline{y_1} \cdots \overline{y_{N+1}} \rangle) \leq N, \quad N \geq 0. \quad (4.16)$$

We first remark that it is enough to consider the above matrix elements, since all of the non-zero matrix elements are included in the above cases. For example, one does not need to consider $\langle \overline{x_1} \cdots \overline{x_N} | A'(z) | \overline{y_1} \cdots \overline{y_{N+1}} \rangle$. This is because due to the so-called ice rule ${}_a \langle \delta | {}_j \langle \gamma | L(z) | {}_a \alpha \rangle {}_j \beta \rangle = 0$ unless $\alpha + \beta = \gamma + \delta$ for the L -operator of the six-vertex model, the A -operators preserve the total number of holes in the quantum space, hence $\langle \overline{x_1} \cdots \overline{x_N} | A'(z) | \overline{y_1} \cdots \overline{y_{N+1}} \rangle = 0$ follows immediately. Similarly, the B -operators increase the total number of holes in the quantum space by one, one can see $\langle \overline{x_1} \cdots \overline{x_N} | B'(z) | \overline{y_1} \cdots \overline{y_N} \rangle = 0$ immediately. It is not difficult to find the above properties of the degree with respect to t' by taking the ice-rule into account.

Let us show Property 1 for the case $N = 1$. We use the decomposition of the double row B -operator (4.10) and insert the completeness relation $\sum_{\overline{y}} |\overline{y}\rangle \langle \overline{y}| = \text{Id}$ between the A - and

B -operators to deform $\det_{t'}(t \langle 1 \cdots M | \mathcal{B}'(z) | \overline{x} \rangle)$ as

$$\begin{aligned} \det_{t'}(t \langle 1 \cdots M | \mathcal{B}'(z) | \overline{x} \rangle) &= \det_{t'}(t \langle 1 \cdots M | (t^{-1} z \tilde{B}'(z) A'(z) + z^{-1} \tilde{A}'(z) B'(z)) | \overline{x} \rangle) \\ &= \deg_{t'}(t^{-1} z \sum_{\overline{y}} \langle 1 \cdots M | \tilde{B}'(z) | \overline{y} \rangle \langle \overline{y} | t A'(z) | \overline{x} \rangle + z^{-1} \langle 1 \cdots M | \tilde{A}'(z) | 1 \cdots M \rangle \langle 1 \cdots M | t B'(z) | \overline{x} \rangle). \end{aligned} \quad (4.17)$$

Using the generic property of the degree $\deg(P+Q) = \text{Max}(\deg P, \deg Q)$ and the properties (4.13), (4.14), (4.15) and (4.16), it follows that $0 \leq \deg_{t'}(t \langle 1 \cdots M | \mathcal{B}'(z) | \overline{x} \rangle) \leq 1$.

Next, let us assume $\deg_{t'}(t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle) \leq N^2$. Let us show $0 \leq \deg_{t'}(t \langle \overline{x_1} \cdots \overline{x_N} | \mathcal{B}'(z) | \overline{y_1} \cdots \overline{y_{N+1}} \rangle) \leq 2N+1$. This can be seen as above by using the decomposition of the double row B -operator (4.10) and inserting the completeness relation

to deform $\deg_{t'}(t\langle \overline{x}_1 \cdots \overline{x}_N | \mathcal{B}'(z) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle)$ as

$$\begin{aligned}
& \deg_{t'}(t\langle \overline{x}_1 \cdots \overline{x}_N | \mathcal{B}'(z) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle) \\
&= \deg_{t'}(t\langle \overline{x}_1 \cdots \overline{x}_N | (t^{-1}z\tilde{B}'(z)A'(z) + z^{-1}\tilde{A}'(z)B'(z)) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle) \\
&= \deg_{t'} \left(t^{-1}z \sum_{\{\overline{u}\}} \langle \overline{x}_1 \cdots \overline{x}_N | \tilde{B}'(z) | \overline{u}_1 \cdots \overline{u}_{N+1} \rangle \langle \overline{u}_1 \cdots \overline{u}_{N+1} | tA'(z) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle \right. \\
& \quad \left. + z^{-1} \sum_{\{\overline{v}\}} \langle \overline{x}_1 \cdots \overline{x}_N | \tilde{A}'(z) | \overline{v}_1 \cdots \overline{v}_N \rangle \langle \overline{v}_1 \cdots \overline{v}_N | tB'(z) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle \right), \tag{4.18}
\end{aligned}$$

and using $\deg(P + Q) = \text{Max}(\deg P, \deg Q)$, the properties (4.13), (4.14), (4.15) and (4.16).

One can finally see $\deg_{t'}(t^{N+1}\langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_{N+1}) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle) \leq (N+1)^2$ by using the property $0 \leq \deg_{t'}(t\langle \overline{x}_1 \cdots \overline{x}_N | \mathcal{B}'(z) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle) \leq 2N+1$ together with the assumption $\deg_{t'}(t^N\langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x}_1 \cdots \overline{x}_N \rangle) \leq N^2$ and the decomposition

$$\begin{aligned}
& t^{N+1}\langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_{N+1}) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle \\
&= \sum_{\{\overline{x}\}} (t^N\langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x}_1 \cdots \overline{x}_N \rangle) (t\langle \overline{x}_1 \cdots \overline{x}_N | \mathcal{B}'(z_{N+1}) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle). \tag{4.19}
\end{aligned}$$

Property 2 can be proved exactly in the same way as in Lemma 2 in Ivanov [43] which is a lengthy one, since one needs many local relations and arguments to prove the lemma. Instead, given below is a shortcut of the argument by transforming his result to Property 2. First, one notes that Ivanov's proof of Lemma 2 shows that not only $\langle x_1 \cdots x_N | \mathcal{B}(z_1) \cdots \mathcal{B}(z_N) | \Omega \rangle$ but also every matrix element $\langle y_1 \cdots y_{n+N} | \mathcal{B}(z_1) \cdots \mathcal{B}(z_N) | x_1 \cdots x_n \rangle$ has the property that $\langle y_1 \cdots y_{n+N} | \mathcal{B}(z_1) \cdots \mathcal{B}(z_N) | x_1 \cdots x_n \rangle / D$ where $D := \prod_{j=1}^N z_j^{j-1-N} (1 + tz_j^2) \prod_{1 \leq j < k \leq N} (1 + tz_j z_k) (1 + tz_j z_k^{-1})$, is invariant under any permutation $z_i \longleftrightarrow z_j$.

From the fact that the rescaled double row B -operator $\mathcal{B}'(z)$ consists of rescaled L -operators and K -matrix, which can be essentially obtained from the original L -operators and K -matrix by the changing the spectral parameters $z \longrightarrow t^{-1}z$, the statement of the invariance for the matrix element $\langle y_1 \cdots y_{n+N} | \mathcal{B}(z_1) \cdots \mathcal{B}(z_N) | x_1 \cdots x_n \rangle$ can be converted to the statement that every matrix element $\langle \overline{x}_1 \cdots \overline{x}_n | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{y}_1 \cdots \overline{y}_{n+N} \rangle$ has the property that $t^N \langle \overline{x}_1 \cdots \overline{x}_n | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{y}_1 \cdots \overline{y}_{n+N} \rangle / D'$ where $D' = \prod_{j=1}^N z_j^{j-1-N} (1 + t'z_j^2) \prod_{1 \leq j < k \leq N} (1 + t'z_j z_k) (t' + z_j z_k^{-1})$, $t' = t^{-1}$, is invariant under any permutation $z_i \longleftrightarrow z_j$. Property 2 follows from this statement since we are considering a special matrix element $t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x}_1 \cdots \overline{x}_N \rangle / D'$.

We prove Property 3 as follows. Since one cannot apply Ivanov's argument as exactly as it is due to the change of the matrix elements of the L -operators, we modify the argument as follows. First, we find that it is enough to show

$$\langle 1 \cdots M | \mathcal{B}'(z) | \overline{x} \rangle = \frac{(1 + t^{-1}z^2)}{tz} \frac{z^{\overline{\lambda}+1} - z^{-\overline{\lambda}-1}}{z - z^{-1}}, \quad \overline{\lambda} = \overline{x} - 1. \tag{4.20}$$

From (4.20) and using the decomposition,

$$\langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x}_1 \cdots \overline{x}_N \rangle = \sum_{\overline{y}} \langle 1 \cdots M | \mathcal{B}'(z_1) | \overline{y} \rangle \langle \overline{y} | \mathcal{B}'(z_2) \cdots \mathcal{B}'(z_N) | \overline{x}_1 \cdots \overline{x}_N \rangle, \tag{4.21}$$

we have

$$\langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle = \frac{(1 + t^{-1} z_1^2)}{t z_1} \sum_{\overline{y}} \frac{z_1^{\overline{y}} - z_1^{-\overline{y}}}{z_1 - z_1^{-1}} \langle \overline{y} | \mathcal{B}'(z_2) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle, \quad (4.22)$$

from which one gets

$$\frac{t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle}{t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle \big|_{z_1 \longleftrightarrow z_1^{-1}}} = \frac{(t' z_1 + z_1^{-1})}{(z_1 + t' z_1^{-1})}. \quad (4.23)$$

This ratio and $\frac{D'}{D' \big|_{z_1 \longleftrightarrow z_1^{-1}}} = \frac{(t' z_1 + z_1^{-1})}{(z_1 + t' z_1^{-1})}$ gives the relation

$$\begin{aligned} & t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle / D' \\ &= t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle / D' \big|_{z_1 \longleftrightarrow z_1^{-1}}, \end{aligned} \quad (4.24)$$

which shows that $t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle / D'$ is invariant under $z_1 \longleftrightarrow z_1^{-1}$. Together with Property 2, one gets Property 3.

Let us show (4.20). We insert the completeness relation to decompose $\langle 1 \cdots M | \mathcal{B}'(z) | \overline{x} \rangle$ as

$$\begin{aligned} & \langle 1 \cdots M | \mathcal{B}'(z) | \overline{x} \rangle \\ &= t^{-1} z \sum_{\overline{y}} \langle 1 \cdots M | \tilde{B}'(z) | \overline{y} \rangle \langle \overline{y} | A'(z) | \overline{x} \rangle + z^{-1} \langle 1 \cdots M | \tilde{A}'(z) | 1 \cdots M \rangle \langle 1 \cdots M | B'(z) | \overline{x} \rangle \\ &= t^{-1} z \sum_{\overline{y}=1, \dots, \overline{x}-1} \langle 1 \cdots M | \tilde{B}'(z) | \overline{y} \rangle \langle \overline{y} | A'(z) | \overline{x} \rangle \\ &+ t^{-1} z \langle 1 \cdots M | \tilde{B}'(z) | \overline{x} \rangle \langle \overline{x} | A'(z) | \overline{x} \rangle + z^{-1} \langle 1 \cdots M | \tilde{A}'(z) | 1 \cdots M \rangle \langle 1 \cdots M | B'(z) | \overline{x} \rangle. \end{aligned} \quad (4.25)$$

Inserting the explicit forms of the matrix elements

$$\langle 1 \cdots M | \tilde{B}'(z) | \overline{y} \rangle = z^{\overline{y}-1}, \quad \overline{y} = 1, \dots, \overline{x} - 1, \quad (4.26)$$

$$\langle \overline{y} | A'(z) | \overline{x} \rangle = \frac{t+1}{t} z^{-\overline{x}+\overline{y}}, \quad \overline{y} = 1, \dots, \overline{x} - 1, \quad (4.27)$$

$$\langle 1 \cdots M | \tilde{B}'(z) | \overline{x} \rangle = z^{\overline{x}-1}, \quad (4.28)$$

$$\langle \overline{x} | A'(z) | \overline{x} \rangle = \frac{1}{t}, \quad (4.29)$$

$$\langle 1 \cdots M | \tilde{A}'(z) | 1 \cdots M \rangle = 1, \quad (4.30)$$

$$\langle 1 \cdots M | B'(z) | \overline{x} \rangle = \frac{1}{t} z^{-\overline{x}+1}, \quad (4.31)$$

into the right hand side of (4.25), we have

$$\begin{aligned}
\langle 1 \cdots M | \mathcal{B}'(z) | \bar{x} \rangle &= t^{-1} z \sum_{\bar{y}=1, \dots, \bar{x}-1} \frac{t+1}{t} z^{-\bar{x}+\bar{y}} z^{\bar{y}-1} + t^{-1} z \frac{1}{t} z^{\bar{x}-1} + z^{-1} \frac{1}{t} z^{-\bar{x}+1} \\
&= t^{-2} (t+1) \frac{z^{\bar{x}-1} - z^{1-\bar{x}}}{z - z^{-1}} + t^{-2} z^{\bar{x}} + t^{-1} z^{-\bar{x}} \\
&= \frac{z + t z^{-1}}{t^2 (z - z^{-1})} (z^{\bar{x}} - z^{-\bar{x}}) = \frac{(1 + t^{-1} z^2)}{t z} \frac{z^{\bar{x}+1} - z^{-\bar{x}-1}}{z - z^{-1}}, \tag{4.32}
\end{aligned}$$

and (4.20) is shown.

Properties 1, 2 and 3 show that $t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \bar{x}_1 \cdots \bar{x}_N \rangle$ is a polynomial of t' with highest degree $2N^2$, and is divided by $D' = \prod_{j=1}^N z_j^{j-1-N} (1 + t' z_j^2) \prod_{1 \leq j < k \leq N} (1 + t' z_j z_k) (t' + z_j z_k^{-1})$. Hence, Lemma 4.2 is proved. \square

From Lemma 4.2, one sees that to study the wavefunction, it is enough to examine a particular value of t . At the point $t = -1$, the six-vertex model reduces to a five-vertex model since the first L -operator becomes

$$L'(z, -1) = -L(-z, -1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix}, \tag{4.33}$$

and second L -operator becomes

$$\tilde{L}'(z, -1) = \tilde{L}(-z, -1) = \begin{pmatrix} -z & 0 & 0 & 0 \\ 0 & z & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.34}$$

The K -matrix at $t = -1$ becomes

$$K'(z, -1) = -K(-z, -1) = \begin{pmatrix} t z & 0 \\ 0 & z^{-1} \end{pmatrix}. \tag{4.35}$$

It is easy to examine at this reduced point $t = -1$, and we find the following relation.

Lemma 4.3. *We have*

$$\begin{aligned}
&\prod_{j=1}^N z_j^{N+1-j} (1 + t^{-1} z_j^2)^{-1} \prod_{1 \leq j < k \leq N} (1 + t^{-1} z_j z_k)^{-1} (t^{-1} + z_j z_k^{-1})^{-1} \\
&\times t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \bar{x}_1 \cdots \bar{x}_N \rangle \Big|_{t=-1} = sp_{\bar{\lambda}}(\{z\}_N). \tag{4.36}
\end{aligned}$$

Proof. From the factorization formula of the determinant

$$\det_N(z_j^{N-k+1} - z_j^{-N+k-1}) = (-1)^N \prod_{j=1}^N z_j^{j-1-N} (1 - z_j^2) \prod_{1 \leq j < k \leq N} (1 - z_j z_k) (1 - z_j z_k^{-1}), \tag{4.37}$$

and the definition of the symplectic Schur functions (3.1), one sees that proving the Lemma is equivalent to showing the following equality

$$\langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle = (-1)^{N(N-1)/2} \det_N(z_j^{\lambda_k + N - k + 1} - z_j^{-\lambda_k - N + k - 1}). \quad (4.38)$$

To show this, let us first list the matrix elements of the single A - and B -operators.

Lemma 4.4. *We have the following explicit expressions for the matrix elements of the A - and B -operators at $t = -1$.*

(1) *The matrix element of $A'(z)$ is given by*

$$\langle \overline{x_1} \cdots \overline{x_k} | A'(z) | \overline{y_1} \cdots \overline{y_k} \rangle = (-1)^k \prod_{j=1}^k \delta_{\overline{x_j y_j}}. \quad (4.39)$$

(2) *The matrix element of $B'(z)$ is given by*

$$\langle \overline{x_1} \cdots \overline{x_{k-1}} | B'(z) | \overline{y_1} \cdots \overline{y_k} \rangle = (-1)^k (-1)^{j-1} z^{1-\overline{y_j}}, \quad (4.40)$$

when the hole configurations $\{\overline{x}\}$ and $\{\overline{y}\}$ satisfy

$\overline{x_1} = \overline{y_1}, \dots, \overline{x_{j-1}} = \overline{y_{j-1}}, \overline{x_j} = \overline{y_{j+1}}, \dots, \overline{x_{k-1}} = \overline{y_k}$ for some j , and 0 otherwise.

(3) *The matrix element of $\tilde{A}'(z)$ is given by*

$$\langle \overline{x_1} \cdots \overline{x_k} | \tilde{A}'(z) | \overline{y_1} \cdots \overline{y_k} \rangle = \prod_{j=1}^k \delta_{\overline{x_j y_j}}, \quad (4.41)$$

(4) *The matrix element of $\tilde{B}'(z)$ is given by*

$$\langle \overline{x_1} \cdots \overline{x_{k-1}} | \tilde{B}'(z) | \overline{y_1} \cdots \overline{y_k} \rangle = (-1)^{j-1} z^{\overline{y_j}-1}, \quad (4.42)$$

when the hole configurations $\{\overline{x}\}$ and $\{\overline{y}\}$ satisfy

$\overline{x_1} = \overline{y_1}, \dots, \overline{x_{j-1}} = \overline{y_{j-1}}, \overline{x_j} = \overline{y_{j+1}}, \dots, \overline{x_{k-1}} = \overline{y_k}$ for some j , and 0 otherwise.

Using these explicit forms of the matrix elements (4.39), (4.40), (4.41) (4.42), and using the decomposition of the double row B -operator (4.10), one finds that the matrix elements of a single double row B -operator $\mathcal{B}'(z)$ at $t = -1$ is given by

$$\langle \overline{x_1} \cdots \overline{x_{k-1}} | \mathcal{B}'(z) | \overline{y_1} \cdots \overline{y_k} \rangle = (-1)^{k+1} (-1)^{j-1} (z^{\overline{y_j}} - z^{-\overline{y_j}}), \quad (4.43)$$

when the hole configurations $\{\overline{x}\}$ and $\{\overline{y}\}$ satisfy

$\overline{x_1} = \overline{y_1}, \dots, \overline{x_{j-1}} = \overline{y_{j-1}}, \overline{x_j} = \overline{y_{j+1}}, \dots, \overline{x_{k-1}} = \overline{y_k}$ for some j , and 0 otherwise.

Since the matrix elements of a single B -operator are essentially the same with the ones for the original wavefunction at $t = -1$ in [43] except the sign $(-1)^{k+1}$ (we also have to translate the hole configurations to particle configurations), the same argument can be applied, and one finds the wavefunction at $t = -1$ is the symplectic Schur functions $\det_N(z_j^{\lambda_k + N - k + 1} - z_j^{\lambda_k - N + k - 1})$ multiplied by a sign factor $\prod_{k=1}^N (-1)^{k+1} = (-1)^{N(N-1)/2}$.

$$\begin{aligned}
& \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle |_{t=-1} \\
&= \prod_{k=1}^N (-1)^{k+1} \sum_{\sigma \in S_N} (-1)^\sigma \prod_{j=1}^N (z_j^{\overline{x_{\sigma(j)}}} - z_j^{-\overline{x_{\sigma(j)}}}) \\
&= \prod_{k=1}^N (-1)^{N(N-1)/2} \sum_{\sigma \in S_N} (-1)^\sigma \prod_{j=1}^N (z_j^{\overline{\lambda_{\sigma(j)}} + N - \sigma(j) + 1} - z_j^{-\overline{\lambda_{\sigma(j)}} - N + \sigma(j) - 1}) \\
&= (-1)^{N(N-1)/2} \det_N (z_j^{\lambda_k + N - k + 1} - z_j^{-\lambda_k - N + k - 1}). \tag{4.44}
\end{aligned}$$

Thus (4.38) is shown, hence (4.36) is proved. \square

Finally, from Lemma 4.2 and (4.36), we have

$$\begin{aligned}
& \prod_{j=1}^N z_j^{N+1-j} (1 + t^{-1} z_j^2)^{-1} \prod_{1 \leq j < k \leq N} (1 + t^{-1} z_j z_k)^{-1} (t^{-1} + z_j z_k^{-1})^{-1} \\
& \times t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle \\
&= \prod_{j=1}^N z_j^{N+1-j} (1 + t^{-1} z_j^2)^{-1} \prod_{1 \leq j < k \leq N} (1 + t^{-1} z_j z_k)^{-1} (t^{-1} + z_j z_k^{-1})^{-1} \\
& \times t^N \langle 1 \cdots M | \mathcal{B}'(z_1) \cdots \mathcal{B}'(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle \Big|_{t=-1} \\
&= sp_{\overline{\lambda}}(\{z\}_N). \tag{4.45}
\end{aligned}$$

which is exactly (4.11), hence Theorem 4.1 is proved. \square

5 A generalization to the factorial symplectic Schur functions

We have showed Theorem 4.1 which gives the relation between the dual wavefunction and the symplectic Schur functions. The proof given in Ivanov [43] and the last section can be lifted to give the exact correspondence between the wavefunction, the dual wavefunction and the factorial symplectic Schur functions by introducing inhomogeneous parameters in the quantum spaces. We state the correspondence in this section.

First we introduce the L -operator of Bump-McNamara-Nakasuji [48] which now has dependence on the quantum space \mathcal{F}_j . The L -operator $L_{aj}(z, t, \alpha_j)$ at the j -th site in the quantum space is given by

$$L_{aj}(z, t, \alpha_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & (t+1)z^{-1} & z^{-1} + \alpha_j & 0 \\ 0 & 0 & 0 & z^{-1} - t\alpha_j \end{pmatrix}. \tag{5.1}$$

The L -operators now have inhomogeneous parameters α_j , $j = 1, \dots, M$ besides the spectral parameters and the deformation parameter. For the case of the wavefunction without

reflecting boundary, these newly introduced inhomogeneous parameters become factorial parameters of the factorial Schur functions in the end.

We also introduce inhomogeneous parameters in the second L -operator. The second L -operator $\tilde{L}_{aj}(z, t, \alpha_j)$ at the j -th site in the quantum space is given by

$$\tilde{L}_{aj}(z, t, \alpha_j) = \begin{pmatrix} z + \alpha_j & 0 & 0 & 0 \\ 0 & tz - \alpha_j & 1 & 0 \\ 0 & (t+1)z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.2)$$

One can also generalize the K -matrix to the following one

$$K_a(z, t, \alpha_0) = \begin{pmatrix} tz - \alpha_0 & 0 \\ 0 & z^{-1} + \alpha_0 \end{pmatrix}, \quad (5.3)$$

where α_0 is a free parameter.

Using these inhomogeneous L -operators and K -matrix, we as again introduce two types of monodromy matrices

$$T_a(z, \{\alpha\}) = L_{aM}(z, t, \alpha_M) \cdots L_{a1}(z, t, \alpha_1), \quad (5.4)$$

and

$$\tilde{T}_a(z, \{\alpha\}) = \tilde{L}_{a1}(z, t, \alpha_1) \cdots \tilde{L}_{aM}(z, t, \alpha_M), \quad (5.5)$$

which act on $W_a \otimes (\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M)$, and denote the matrix elements of the two monodromy matrices as

$$A(z, \{\alpha\}) = {}_a\langle 0 | T_a(z, \{\alpha\}) | 0 \rangle_a, \quad (5.6)$$

$$B(z, \{\alpha\}) = {}_a\langle 0 | T_a(z, \{\alpha\}) | 1 \rangle_a, \quad (5.7)$$

and

$$\tilde{A}(z, \{\alpha\}) = {}_a\langle 1 | \tilde{T}_a(z, \{\alpha\}) | 1 \rangle_a, \quad (5.8)$$

$$\tilde{B}(z, \{\alpha\}) = {}_a\langle 0 | \tilde{T}_a(z, \{\alpha\}) | 1 \rangle_a. \quad (5.9)$$

Here, $\{\alpha\} = \{\alpha_1, \dots, \alpha_M\}$ is included in the notation to indicate that the operators depend on this set of parameters. As again, we introduce the following double row B -operator.

$$\mathcal{B}(z, \{\bar{\alpha}\}) = \tilde{B}(z, \{\alpha\})_a \langle 0 | K(z, t, \alpha_0) | 0 \rangle_a A(z) + \tilde{A}(z, \{\alpha\})_a \langle 1 | K(z, t, \alpha_0) | 1 \rangle_a B(z, t, \{\alpha\}) \quad (5.10)$$

$$= (tz - \alpha_0) \tilde{B}(z, \{\alpha\}) A(z, \{\alpha\}) + (z^{-1} + \alpha_0) \tilde{A}(z, \{\alpha\}) B(z, \{\alpha\}). \quad (5.11)$$

Since the double row B -operator uses the generalized K -matrix as a component, the dependence on the inhomogeneous parameters is lifted to the set of parameters $\{\bar{\alpha}\} = \{\alpha_0, \alpha_1, \dots, \alpha_M\}$ where α_0 is added to $\{\alpha\}$.

We now introduce the inhomogeneous wavefunction $\langle x_1 \cdots x_N | \Psi(z_1, \dots, z_N, \{\alpha\}) \rangle$ as the overlap between the particle configurations $\langle x_1 \cdots x_N |$ and the inhomogeneous N -particle state

$$\Psi(z_1, \dots, z_N, \{\bar{\alpha}\}) = \mathcal{B}(z_1, \{\bar{\alpha}\}) \cdots \mathcal{B}(z_N, \{\bar{\alpha}\}) | \Omega \rangle. \quad (5.12)$$

Likewise, the dual wavefunction $\langle \Phi(z_1, \dots, z_N, \{\alpha\}) | \overline{x_1} \cdots \overline{x_N} \rangle$ is introduced as the overlap between the hole configurations $|\overline{x_1} \cdots \overline{x_N}\rangle$ and the inhomogeneous dual N -particle state

$$\langle \Phi(z_1, \dots, z_N, \{\overline{\alpha}\}) | = \langle 1 \cdots M | \mathcal{B}(z_1, \{\overline{\alpha}\}) \cdots \mathcal{B}(z_N, \{\overline{\alpha}\}). \quad (5.13)$$

These wavefunctions can be expressed by the factorial symplectic Schur functions defined below.

Definition 5.1. The factorial symplectic Schur functions is defined to be the following determinant:

$$sp_\lambda(\{z\}_N | \{\overline{\alpha}\}) = \frac{G_{\lambda+\delta}(\{z\}_N | \{\overline{\alpha}\})}{\det_N(z_j^{N-k+1} - z_j^{-N+k-1})}, \quad (5.14)$$

where $\{z\}_N = \{z_1, \dots, z_N\}$ is a set of variables and λ denotes a Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ with weakly decreasing non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$, and $\delta = (N-1, N-2, \dots, 0)$. $G_\mu(\{z\}_N | \{\overline{\alpha}\})$ is an $N \times N$ determinant

$$G_\mu(\{z\}_N | \{\overline{\alpha}\}) = \det_N \left(\prod_{j=0}^{\mu} (z_k + \alpha_j) - \prod_{j=0}^{\mu} (z_k^{-1} + \alpha_j) \right). \quad (5.15)$$

We remark that one must respect the ordering of the factorial parameters $\{\overline{\alpha}\} = \{\alpha_0, \alpha_1, \dots, \alpha_M\}$.

We have the following correspondence between the wavefunction of the free-fermion model with inhomogeneties and the factorial Schur symplectic functions.

Theorem 5.2. *The wavefunction $\langle x_1 \cdots x_N | \Psi(z_1, \dots, z_N, \{\overline{\alpha}\}) \rangle$ is expressed by the factorial symplectic functions as*

$$\begin{aligned} & \langle x_1 \cdots x_N | \Psi(z_1, \dots, z_N, \{\overline{\alpha}\}) \rangle \\ &= \prod_{j=1}^N z_j^{j-1-N} (1 + tz_j^2) \prod_{1 \leq j < k \leq N} (1 + tz_j z_k) (1 + tz_j z_k^{-1}) sp_\lambda(\{z\}_N | \{\overline{\alpha}\}), \end{aligned} \quad (5.16)$$

under the relation $\lambda_j = x_{N-j+1} - N + j - 1$, $j = 1, \dots, N$.

This Theorem can be proved by noting that the arguments in Ivanov [43] naturally lift to this inhomogeneous setting. One first shows that the wavefunction is a polynomial of t with highest degree N^2 whose t -dependent part can be factorized as $\prod_{j=1}^N (1 + tz_j^2) \prod_{1 \leq j < k \leq N} (1 + tz_j z_k) (1 + tz_j z_k^{-1})$. Then one evaluates the wavefunction at $t = -1$, at which the six-vertex model reduces to a five-vertex model, and each configuration making non-zero contribution ($2^N \times N!$ configurations in total) to the wavefunction essentially corresponds to each term of the determinant expansion of the numerator of the factorial symplectic Schur functions (5.14).

As for the inhomogeneous dual wavefunction, one can apply the argument in section 4 and get the following relation with the factorial symplectic Schur functions.

Theorem 5.3. *The dual wavefunction $\langle \Phi(z_1, \dots, z_N, \{\bar{\alpha}\}) | \bar{x}_1 \cdots \bar{x}_N \rangle$ can be expressed by the factorial symplectic Schur functions as*

$$\begin{aligned} & \langle \Phi(z_1, \dots, z_N, \{\bar{\alpha}\}) | \bar{x}_1 \cdots \bar{x}_N \rangle \\ &= t^{N(M-N)} \prod_{j=1}^N z_j^{j-1-N} (1 + tz_j^2) \prod_{1 \leq j < k \leq N} (1 + tz_j z_k) (1 + tz_j z_k^{-1}) sp_{\overline{\lambda}}(\{tz\}_N | \{-\bar{\alpha}\}). \end{aligned} \quad (5.17)$$

Here the Young diagram for the factorial symplectic Schur functions corresponds to the hole configuration under the relation $\bar{\lambda}_j = \bar{x}_{N-j+1} - N + j - 1$, $j = 1, \dots, N$, and the symmetric variables are $\{tz\}_N = \{tz_1, \dots, tz_N\}$. Moreover, the signs of the parameters of the factorial symplectic Schur functions in the right hand side of (5.17) are now inverted simultaneously: $\{-\bar{\alpha}\} = \{-\alpha_0, -\alpha_1, \dots, -\alpha_M\}$.

The correspondence (5.17) can be proved by naturally lifting the arguments given in section 4 to this inhomogeneous setting. At the point when $t = -1$ where the six-vertex model reduces to the five-vertex model, the introduction of inhomogeneous parameters is reflected in the t -independent part of the correspondence. The right hand side of the final expression of the correspondence in (5.17) is lifted to the factorial symplectic Schur functions.

6 Conclusion

We investigated the free-fermion model under the reflecting boundary condition, and showed the precise relation between the dual wavefunction and the symplectic Schur functions. The result and the proof is an extension of the ones in [47] for the case without reflecting boundary, where the statement was transformed into another equivalent one so that one can use the arguments given by [26] and [43]. The correspondence can be regarded as a type C version of the dual version of the Tokuyama formula by Ivanov.

In its relation with automorphic representation theory, the wavefunction with another boundary K -matrices is introduced by Brubaker-Bump-Chinta-Gunnells [44]. The conjectures about the correspondence may be solved by using other ideas such as the theory of divided difference operators. As for the wavefunction with reflecting boundary condition, we remark that there are several works on the XXZ model with reflecting boundary condition and its degeneration in [54, 55, 56, 57, 58], for example.

We also generalized the correspondence between the wavefunction, the dual wavefunction and the symplectic Schur functions to factorial symplectic Schur functions by using the first inhomogeneous L -operator in [48], the second inhomogeneous L -operator and the inhomogeneous K -matrix. The result is a symplectic version of the result in [48], where the correspondence between the wavefunction without reflecting boundary and the factorial Schur polynomials is established. We extended furthermore to the free-fermion model with two types of inhomogeneous parameters, and there are correspondences between the original and the dual wavefunctions and a generalization of the factorial Schur polynomials and factorial symplectic Schur functions [59, 60]. Details will appear elsewhere. Recently, the Tokuyama-type formula for classical groups were realized combinatorially using the methods of non-intersecting lattice paths by Hamel-King [41, 42]. Factorial characters also appear in their work. It is worthwhile studying the relation with integrable models, which may lead to further studies on the subject.

We finally remark that number theorists regard the free-fermion six-vertex model as a special case of the “metaplectic ice” (see [33] for example). Recently, they established the relation between the Yang-Baxter equation for the Perk-Schultz model and the metaplectic ice [32]. It seems worthwhile to study these models and find novel combinatorial formulas by means of modern statistical physical methods and techniques developed to analyze quantum integrable models.

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A Matrix elements

We first list the matrix elements of the A - and B -operators.

Proposition A.1. (1) *The matrix elements of a single A -operator $A(z)$ is given by*

$$\begin{aligned} \langle \bar{x}_1 \cdots \bar{x}_N | A(z) | \bar{y}_1 \cdots \bar{y}_N \rangle = & (t+1)^{|\{\bar{x}_j, j=1, \dots, N \mid \bar{x}_j \neq \bar{y}_j, \bar{x}_j \neq \bar{y}_{j-1}\}|} \\ & \times t^{\sum_{j=0}^N \text{Max}(\bar{x}_{j+1} - \bar{y}_j - 1, 0)} z^{\sum_{j=1}^N (\bar{x}_j - \bar{y}_j)}, \end{aligned} \quad (\text{A.1})$$

for hole configurations $\{\bar{x}\}$ ($1 \leq \bar{x}_1 < \cdots < \bar{x}_N \leq M$) and $\{\bar{y}\}$ ($1 \leq \bar{y}_1 < \cdots < \bar{y}_N \leq M$) satisfying the interlacing relation $\bar{x}_1 \leq \bar{y}_1 \leq \bar{x}_2 \leq \cdots \leq \bar{x}_N \leq \bar{y}_N$, and 0 otherwise. Here we also set $\bar{y}_0 = 0$ and $\bar{x}_{N+1} = M+1$.

(2) *The matrix elements of a single B -operator $B(z)$ is given by*

$$\begin{aligned} \langle \bar{x}_1 \cdots \bar{x}_N | B(z) | \bar{y}_1 \cdots \bar{y}_{N+1} \rangle = & (t+1)^{|\{\bar{x}_j, j=1, \dots, N \mid \bar{x}_j \neq \bar{y}_j, \bar{x}_j \neq \bar{y}_{j+1}\}|} \\ & \times t^{\sum_{j=1}^{N+1} \text{Max}(\bar{x}_j - \bar{y}_j - 1, 0)} z^{\sum_{j=1}^N (\bar{x}_j - \bar{y}_{j+1})}, \end{aligned} \quad (\text{A.2})$$

for hole configurations $\{\bar{x}\}$ ($1 \leq \bar{x}_1 < \cdots < \bar{x}_N \leq M$) and $\{\bar{y}\}$ ($1 \leq \bar{y}_1 < \cdots < \bar{y}_{N+1} \leq M$) satisfying the interlacing relation $\bar{y}_1 \leq \bar{x}_1 \leq \bar{y}_2 \leq \bar{x}_2 \leq \cdots \leq \bar{x}_N \leq \bar{y}_{N+1}$, and 0 otherwise. Here we also set $\bar{x}_{N+1} = M+1$.

(3) *The matrix elements of a single \tilde{A} -operator $\tilde{A}(z)$ is given by*

$$\begin{aligned} \langle \bar{x}_1 \cdots \bar{x}_N | \tilde{A}(z) | \bar{y}_1 \cdots \bar{y}_N \rangle = & (t+1)^{|\{\bar{x}_j, j=1, \dots, N \mid \bar{x}_j \neq \bar{y}_j, \bar{x}_j \neq \bar{y}_{j-1}\}|} \\ & \times t^{\sum_{j=1}^N \text{Max}(\bar{y}_j - \bar{x}_j - 1, 0)} z^{\sum_{j=1}^N (\bar{y}_j - \bar{x}_j)}, \end{aligned} \quad (\text{A.3})$$

for hole configurations $\{\bar{x}\}$ ($1 \leq \bar{x}_1 < \cdots < \bar{x}_N \leq M$) and $\{\bar{y}\}$ ($1 \leq \bar{y}_1 < \cdots < \bar{y}_N \leq M$) satisfying the interlacing relation $\bar{x}_1 \leq \bar{y}_1 \leq \bar{x}_2 \leq \cdots \leq \bar{x}_N \leq \bar{y}_N$, and 0 otherwise. We also set $\bar{y}_0 = 0$.

(4) *The matrix elements of a single \tilde{B} -operator $\tilde{B}(z)$ is given by*

$$\begin{aligned} \langle \bar{x}_1 \cdots \bar{x}_N | \tilde{B}(z) | \bar{y}_1 \cdots \bar{y}_{N+1} \rangle = & (t+1)^{|\{\bar{x}_j, j=1, \dots, N \mid \bar{x}_j \neq \bar{y}_j, \bar{x}_j \neq \bar{y}_{j+1}\}|} \\ & \times t^{\sum_{j=1}^{N+1} \text{Max}(\bar{y}_j - \bar{x}_{j-1} - 1, 0)} z^{\sum_{j=1}^{N+1} (\bar{y}_j - \bar{x}_{j-1}) - 1}, \end{aligned} \quad (\text{A.4})$$

for hole configurations $\{\bar{x}\}$ ($1 \leq \bar{x}_1 < \cdots < \bar{x}_N \leq M$) and $\{\bar{y}\}$ ($1 \leq \bar{y}_1 < \cdots < \bar{y}_{N+1} \leq M$) satisfying the interlacing relation $\bar{y}_1 \leq \bar{x}_1 \leq \bar{y}_2 \leq \bar{x}_2 \leq \cdots \leq \bar{x}_N \leq \bar{y}_{N+1}$, and 0 otherwise. Here we also set $\bar{x}_0 = 0$.

Proof. The matrix elements (A.2) is essentially calculated in [47] (we just need to reverse the signs of the powers of z to get the result (A.2) because of the change of the definition of the L -operator (2.8)). The other cases (A.1), (A.3) and (A.4) can be calculated in the same way as in [47].

Let us show (A.4) for example. Let us first count the powers of the spectral parameter z . If the hole configurations $\{\bar{x}\}$ and $\{\bar{y}\}$ are fixed and satisfies the interlacing relation $\bar{y}_1 \leq \bar{x}_1 \leq \bar{y}_2 \leq \bar{x}_2 \leq \dots \leq \bar{x}_N \leq \bar{y}_{N+1}$, the inner states in the auxiliary space is fixed uniquely, which is a sequence of 0's and 1's. We observe that for each sequence $10 \dots 01$ of the inner states in the auxiliary space, all the matrix elements of the L -operators (2.16) in between contribute to the power z , and gives $z^{\sum_j (\bar{y}_j - \bar{x}_{j-1})}$ for some sum over j . Taking all of the $01 \dots 10$ sequences into account, we have the factor $z^{\sum_{j=1}^{N+1} (\bar{y}_j - \bar{x}_{j-1}) - 1}$. Here, we also take into account the first sequence consisting only of 0's $0 \dots 0$, which contribute to the factor $z^{\bar{y}_1 - \bar{x}_0 - 1}$.

Let us turn to count the powers of $t + 1$ and t . We get a factor $t + 1$ for each case when both $\bar{x}_j \neq \bar{y}_j$ and $\bar{x}_j \neq \bar{y}_{j+1}$ are satisfied since the matrix element of the L -operator is $[L(z, t)]_{01}^{10} = (t+1)z$ at the \bar{x}_j -th site for this case. One gets $(t+1)^{|\{\bar{x}_j, j=1, \dots, N \mid \bar{x}_j \neq \bar{y}_j, \bar{x}_j \neq \bar{y}_{j+1}\}|}$ in total.

Next, we count the powers of t . If $\bar{y}_j < \bar{x}_j$ is satisfied, the matrix elements of the L -operators are all $[L(z, t)]_{01}^{01} = tz$ from the $(\bar{x}_{j-1} + 1)$ -th site to the $(\bar{y}_j - 1)$ -th site. On the other hand, $[L(z, t)]_{01}^{01}$ does not appear if $\bar{x}_{j-1} = \bar{y}_j$, and there is no contribution to the power of t for this case. The contributions from t is given by $t^{\sum_{j=1}^{N+1} \text{Max}(\bar{y}_j - \bar{x}_{j-1} - 1, 0)}$.

Having calculated all factors, one finds the matrix elements are given by (A.4) □

Example of $\langle \bar{x}_1 \dots \bar{x}_N | A(z) | \bar{y}_1 \dots \bar{y}_N \rangle$

Let $M = 15$, $N = 4$, $\bar{x} = (3, 5, 8, 11)$ and $\bar{y} = (3, 6, 11, 13)$. We also set $\bar{x}_5 = 15 + 1 = 16$ and $\bar{y}_0 = 0$. From $\text{Max}(\bar{x}_1 - \bar{y}_0 - 1, 0) = \text{Max}(3 - 0 - 1, 0) = 2$, $\text{Max}(\bar{x}_2 - \bar{y}_1 - 1, 0) = \text{Max}(5 - 3 - 1, 0) = 1$, $\text{Max}(\bar{x}_3 - \bar{y}_2 - 1, 0) = \text{Max}(8 - 6 - 1, 0) = 1$, $\text{Max}(\bar{x}_4 - \bar{y}_3 - 1, 0) = \text{Max}(11 - 11 - 1, 0) = 0$, $\text{Max}(\bar{x}_5 - \bar{y}_4 - 1, 0) = \text{Max}(16 - 13 - 1, 0) = 2$, we have the factor $t^{2+1+1+0+2} = t^6$. The relations $\bar{y}_0 \neq \bar{x}_1 = \bar{y}_1$, $\bar{y}_1 \neq \bar{x}_2 \neq \bar{y}_2$, $\bar{y}_2 \neq \bar{x}_3 \neq \bar{y}_3$, $\bar{y}_3 = \bar{x}_4 \neq \bar{y}_4$ give the factor $(t+1)^2$, and we also have the factor z^{-6} from $(\bar{x}_1 - \bar{y}_1) + (\bar{x}_2 - \bar{y}_2) + (\bar{x}_3 - \bar{y}_3) + (\bar{x}_4 - \bar{y}_4) = (3 - 3) + (5 - 6) + (8 - 11) + (11 - 13) = 0 - 1 - 3 - 2 = -6$. In total, the right hand side of (A.1) is calculated as $(t+1)^2 t^6 z^{-6}$. One can check that this matches the left hand side of (A.1), i.e., the matrix elements of the corresponding A -operator by explicit calculation.

Example of $\langle \bar{x}_1 \dots \bar{x}_N | B(z) | \bar{y}_1 \dots \bar{y}_{N+1} \rangle$

Let $M = 10$, $N = 2$, $\bar{x} = (3, 6)$ and $\bar{y} = (1, 6, 8)$. We also set $\bar{x}_3 = 10 + 1 = 11$. From $\text{Max}(\bar{x}_1 - \bar{y}_1 - 1, 0) = \text{Max}(3 - 1 - 1, 0) = 1$, $\text{Max}(\bar{x}_2 - \bar{y}_2 - 1, 0) = \text{Max}(6 - 6 - 1, 0) = 0$, $\text{Max}(\bar{x}_3 - \bar{y}_3 - 1, 0) = \text{Max}(11 - 8 - 1, 0) = 2$, we have the factor $t^{1+0+2} = t^3$. The relations $\bar{y}_1 \neq \bar{x}_1 \neq \bar{y}_2$, $\bar{y}_2 = \bar{x}_2 \neq \bar{y}_3$ give the factor $(t+1)^1 = t+1$, and we also have the factor z^{-5} from $(\bar{x}_1 - \bar{y}_2) + (\bar{x}_2 - \bar{y}_3) = (3 - 6) + (6 - 8) = -3 - 2 = -5$. In total, the right hand side of (A.2) is calculated as $(t+1)t^3 z^{-5}$. One can check that this matches the left hand side of (A.2), i.e., the matrix elements of the corresponding B -operator by explicit calculation.

Example of $\langle \bar{x}_1 \dots \bar{x}_N | \tilde{A}(z) | \bar{y}_1 \dots \bar{y}_N \rangle$

Let $M = 15$, $N = 4$, $\bar{x} = (2, 5, 10, 13)$ and $\bar{y} = (2, 8, 10, 15)$. We also set $\bar{y}_0 = 0$. From

$\text{Max}(\overline{y}_1 - \overline{x}_1 - 1, 0) = \text{Max}(2 - 2 - 1, 0) = 0$, $\text{Max}(\overline{y}_2 - \overline{x}_2 - 1, 0) = \text{Max}(8 - 5 - 1, 0) = 2$, $\text{Max}(\overline{y}_3 - \overline{x}_3 - 1, 0) = \text{Max}(10 - 10 - 1, 0) = 0$, $\text{Max}(\overline{y}_4 - \overline{x}_4 - 1, 0) = \text{Max}(15 - 13 - 1, 0) = 1$, we have the factor $t^{0+2+0+1} = t^3$. The relations $\overline{y}_0 \neq \overline{x}_1 = \overline{y}_1$, $\overline{y}_1 \neq \overline{x}_2 \neq \overline{y}_2$, $\overline{y}_2 \neq \overline{x}_3 = \overline{y}_3$, $\overline{y}_3 \neq \overline{x}_4 \neq \overline{y}_4$ give the factor $(t+1)^2$, and we also have the factor z^5 from $(\overline{y}_1 - \overline{x}_1) + (\overline{y}_2 - \overline{x}_2) + (\overline{y}_3 - \overline{x}_3) + (\overline{y}_4 - \overline{x}_4) = (2-2) + (8-5) + (10-10) + (15-13) = 0+3+0+2 = 5$. In total, the right hand side of (A.3) is calculated as $(t+1)^2 t^3 z^5$. One can check that this matches the left hand side of (A.3), i.e., the matrix elements of the corresponding A -operator by explicit calculation.

Example of $\langle \overline{x}_1 \cdots \overline{x}_N | \tilde{B}(z) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle$

Let $M = 10$, $N = 2$, $\overline{x} = (5, 8)$ and $\overline{y} = (3, 5, 10)$. We also set $\overline{x}_0 = 0$. From $\text{Max}(\overline{y}_1 - \overline{x}_0 - 1, 0) = \text{Max}(3 - 0 - 1, 0) = 2$, $\text{Max}(\overline{y}_2 - \overline{x}_1 - 1, 0) = \text{Max}(5 - 5 - 1, 0) = 0$, $\text{Max}(\overline{y}_3 - \overline{x}_2 - 1, 0) = \text{Max}(10 - 8 - 1, 0) = 1$, we have the factor $t^{2+0+1} = t^3$. The relations $\overline{y}_1 \neq \overline{x}_1 = \overline{y}_2$, $\overline{y}_2 \neq \overline{x}_2 \neq \overline{y}_3$ give the factor $(t+1)^1 = t+1$, and we also have the factor z^4 from $(\overline{y}_1 - \overline{x}_0) + (\overline{y}_2 - \overline{x}_1) + (\overline{y}_3 - \overline{x}_2) - 1 = (3-0) + (5-5) + (10-8) - 1 = 3+0+2-1 = 4$. In total, the right hand side of (A.4) is calculated as $(t+1)t^3 z^4$. One can check that this matches the left hand side of (A.4), i.e., the matrix elements of the corresponding B -operator by explicit calculation (see Figure 12 for a graphical description of the corresponding matrix element).

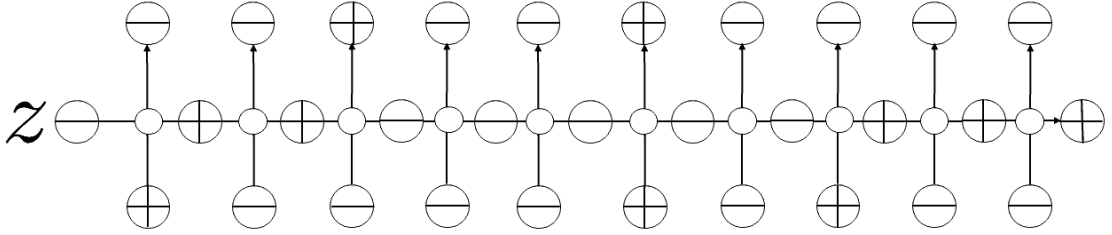


Figure 12: The matrix element $\langle \overline{x}_1 \cdots \overline{x}_N | \tilde{B}(z) | \overline{y}_1 \cdots \overline{y}_{N+1} \rangle$ for $M = 10$, $N = 2$, $\overline{x} = (5, 8)$ and $\overline{y} = (3, 5, 10)$. One sees that the inner state is uniquely fixed, and the matrix element is calculated by multiplying the matrix elements of the L -operators $1 \times tz \times (t+1)z \times 1 \times 1 \times 1 \times 1 \times 1 \times tz \times tz = (t+1)t^3 z^4$.

Combining the matrix elements of the single A - and B -operators (A.1), (A.2), (A.3) and (A.4), one can calculate the matrix elements of the double row B -operators.

Proposition A.2. *The matrix elements of the double row B-operator is given by*

$$\langle \overline{x_N} \cdots \overline{x_1} | \mathcal{B}(z) | \overline{x_1^{N+1}} \cdots \overline{x_{N+1}^{N+1}} \rangle = \alpha(z, \{\overline{x^N}\}, \{\overline{x^{N+1}}\}) + \beta(z, \{\overline{x^N}\}, \{\overline{x^{N+1}}\}), \quad (\text{A.5})$$

for hole configurations $\{\overline{x^N}\}$ ($1 \leq \overline{x_1^N} < \cdots < \overline{x_N^N} \leq M$) and $\{\overline{x^{N+1}}\}$ ($1 \leq \overline{x_1^{N+1}} < \cdots < \overline{x_{N+1}^{N+1}} \leq M$) satisfying the interlacing relation $\overline{x_1^{N+1}} \leq \overline{x_1^N} \leq \overline{x_2^{N+1}} \leq \overline{x_2^N} \leq \cdots \leq \overline{x_N^N} \leq \overline{x_{N+1}^{N+1}}$, and 0 otherwise.

$\alpha(z, \{\overline{x^N}\}, \{\overline{x^{N+1}}\})$ is given by

$$\begin{aligned} & \alpha(z, \{\overline{x^N}\}, \{\overline{x^{N+1}}\}) \\ &= \sum_{\{\overline{y^N}\}} (t+1)^{|\{\overline{x_j^N}, j=1, \dots, N \mid \overline{x_j^N} \neq \overline{y_j^N}, \overline{x_j^N} \neq \overline{y_{j-1}^N}\}| + |\{\overline{y_j^N}, j=1, \dots, N \mid \overline{y_j^N} \neq \overline{x_j^{N+1}}, \overline{y_j^N} \neq \overline{x_{j+1}^{N+1}}\}|} \\ & \quad \times t^{\sum_{j=1}^N \text{Max}(\overline{y_j^N} - \overline{x_j^N} - 1, 0) + \sum_{j=1}^{N+1} \text{Max}(\overline{y_j^N} - \overline{x_j^{N+1}} - 1, 0)} \\ & \quad \times z^{\sum_{j=1}^N (\overline{y_j^N} - \overline{x_j^N}) + \sum_{j=1}^N (\overline{y_j^N} - \overline{x_{j+1}^{N+1}}) - 1}, \end{aligned} \quad (\text{A.6})$$

where we have set $\overline{y_{N+1}^N} = M+1$, $\overline{y_0^N} = 0$, and take the sum over $\{\overline{y^N}\} = \{\overline{y_1^N}, \dots, \overline{y_N^N}\}$ such that $\text{Max}(\overline{x_j^N}, \overline{x_j^{N+1}}) \leq \overline{y_j^N} \leq \text{Min}(\overline{x_{j+1}^N}, \overline{x_{j+1}^{N+1}})$ is satisfied for each $j = 1, \dots, N$.

$\beta(z, \{\overline{x^N}\}, \{\overline{x^{N+1}}\})$ is given by

$$\begin{aligned} & \beta(z, \{\overline{x^N}\}, \{\overline{x^{N+1}}\}) \\ &= \sum_{\{\overline{y^N}\}} (t+1)^{|\{\overline{x_j^N}, j=1, \dots, N \mid \overline{x_j^N} \neq \overline{y_j^N}, \overline{x_j^N} \neq \overline{y_{j+1}^N}\}| + |\{\overline{y_j^N}, j=1, \dots, N+1 \mid \overline{y_j^N} \neq \overline{x_j^{N+1}}, \overline{y_j^N} \neq \overline{x_{j-1}^{N+1}}\}|} \\ & \quad \times t^{\sum_{j=1}^{N+1} \text{Max}(\overline{y_j^N} - \overline{x_{j-1}^N} - 1, 0) + \sum_{j=0}^{N+1} \text{Max}(\overline{y_{j+1}^N} - \overline{x_j^{N+1}} - 1, 0) + 1} \\ & \quad \times z^{\sum_{j=1}^{N+1} (\overline{y_j^N} - \overline{x_{j-1}^N}) + \sum_{j=1}^{N+1} (\overline{y_j^N} - \overline{x_j^{N+1}})}, \end{aligned} \quad (\text{A.7})$$

where we have set $\overline{y_{N+2}^N} = M+1$, and take the sum over $\{\overline{y^N}\} = \{\overline{y_1^N}, \dots, \overline{y_N^N}\}$ such that $\text{Max}(\overline{x_{j-1}^N}, \overline{x_{j-1}^{N+1}}) \leq \overline{y_j^N} \leq \text{Min}(\overline{x_j^N}, \overline{x_j^{N+1}})$ is satisfied for each $j = 1, \dots, N$. Here we also set $\overline{x_0^N} = \overline{x_0^{N+1}} = 0$.

$\alpha(z, \{\overline{x^N}\}, \{\overline{x^{N+1}}\})$ and $\beta(z, \{\overline{x^N}\}, \{\overline{x^{N+1}}\})$ are the explicit forms of the matrix elements $\langle \overline{x_N} \cdots \overline{x_1} | z^{-1} \tilde{A}(z) B(z) | \overline{x_1^{N+1}} \cdots \overline{x_{N+1}^{N+1}} \rangle$ and $\langle \overline{x_N} \cdots \overline{x_1} | t z \tilde{B}(z) A(z) | \overline{x_1^{N+1}} \cdots \overline{x_{N+1}^{N+1}} \rangle$ which are calculated by inserting the completeness relation between $\tilde{A}(z)$ and $B(z)$ and using (A.2) (A.3), and by inserting the completeness relation between $\tilde{B}(z)$ and $A(z)$ and using (A.1) (A.4), respectively. The sum of those two matrix elements becomes the explicit form of the matrix elements $\langle \overline{x_N} \cdots \overline{x_1} | \mathcal{B}(z) | \overline{x_1^{N+1}} \cdots \overline{x_{N+1}^{N+1}} \rangle$ due to the decomposition of the double row B-operator (2.33).

Inserting the completeness relation between each double row B-operators and using (A.5) repeatedly, one gets the explicit form of the dual wavefunction $\langle \Phi(z_1, \dots, z_N) | \overline{x_1^N} \cdots \overline{x_N^N} \rangle$.

Proposition A.3. *The matrix elements of the dual wavefunction $\langle \Phi(z_1, \dots, z_N) | \overline{x_1^N} \cdots \overline{x_N^N} \rangle$ is given by*

$$\langle \Phi(z_1, \dots, z_N) | \overline{x_1^N} \cdots \overline{x_N^N} \rangle = \sum_{\{\overline{x^1}\}, \dots, \{\overline{x^{N-1}}\}} \prod_{k=1}^N (\alpha(z_k, \{\overline{x^{k-1}}\}, \{\overline{x^k}\}) + \beta(z_k, \{\overline{x^{k-1}}\}, \{\overline{x^k}\})), \quad (\text{A.8})$$

where the sum is over all sequences of hole configurations $\{\overline{x^k}\}$, $k = 1, \dots, N-1$ ($1 \leq \overline{x_1^k} < \cdots < \overline{x_k^k} \leq M$) satisfying the interlacing relations $\overline{x_1^{k+1}} \leq \overline{x_1^k} \leq \overline{x_2^{k+1}} \leq \overline{x_2^k} \leq \cdots \leq \overline{x_k^{k+1}} \leq \overline{x_{k+1}^{k+1}}$ for all $k = 1, \dots, N-1$. $\alpha(z, \{\overline{x^{k-1}}\}, \{\overline{x^k}\})$ and $\beta(z, \{\overline{x^{k-1}}\}, \{\overline{x^k}\})$ are given by (A.6) and (A.7).

Combining the above result with (4.3) in Theorem 4.1, one gets the following.

Corollary A.4. *The following identity holds*

$$\begin{aligned} & t^{N(M-N)} \prod_{j=1}^N z_j^{j-1-N} (1 + tz_j^2) \prod_{1 \leq j < k \leq N} (1 + tz_j z_k) (1 + tz_j z_k^{-1}) sp_{\overline{\lambda}}(\{tz\}_N) \\ &= \sum_{\{\overline{x^1}\}, \dots, \{\overline{x^{N-1}}\}} \prod_{k=1}^N (\alpha(z_k, \{\overline{x^{k-1}}\}, \{\overline{x^k}\}) + \beta(z_k, \{\overline{x^{k-1}}\}, \{\overline{x^k}\})), \end{aligned} \quad (\text{A.9})$$

where $\alpha(z, \{\overline{x^{k-1}}\}, \{\overline{x^k}\})$ and $\beta(z, \{\overline{x^{k-1}}\}, \{\overline{x^k}\})$ are given by (A.6) and (A.7), and the sum is over all sequences of hole configurations $\{\overline{x^k}\}$, $k = 1, \dots, N-1$ ($1 \leq \overline{x_1^k} < \cdots < \overline{x_k^k} \leq M$) satisfying the interlacing relations $\overline{x_1^{k+1}} \leq \overline{x_1^k} \leq \overline{x_2^{k+1}} \leq \overline{x_2^k} \leq \cdots \leq \overline{x_k^{k+1}} \leq \overline{x_{k+1}^{k+1}}$ for all $k = 1, \dots, N-1$. $\overline{x_j^N}$, $j = 1, \dots, N$ are uniquely fixed by $\overline{\lambda_j}$, $j = 1, \dots, N$ under the relation $\overline{x_j^N} = \overline{\lambda_{N+1-j}} + j$, $j = 1, \dots, N$.

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